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Lines of Curvature on Annular Surfaces having Two Spherical Directrices.

By Virgil Snyder.

Let $x_1, x_2, \ldots, x_6$ be any six numbers which satisfy the quadratic identity

$$\Pi(x_1, x_2, \ldots, x_6) \equiv \Pi(x) \equiv 0\,;$$

these numbers will be taken as the homogeneous coordinates of a sphere. Suppose these coordinates to be functions of one parameter t; the spheres obtained by giving t all values in succession envelop an annular surface; the circle of intersection of two consecutive spheres lies on the surface and is a line of curvature. An annular surface may be defined as a surface having one system of lines of curvature composed of circles.

Consider the spheres which touch two consecutive spheres of the surface: these spheres constitute a special linear congruence; they must touch the sphere x in the points of its circle of intersection c with the consecutive sphere $x + dx$. The centres of the spheres of this congruence lie on a cone of revolution determined by c and the centre of x.

In case two consecutive spheres of the surface touch each other, their circle of intersection reduces to a point. The congruence is now degraded; it consists of the two hyperpencils of spheres containing one or the other of the two minimum lines determined by the point of tangency and the common tangent plane. On account of the close analogy between these points and pinch-points on ruled surfaces, I will apply the name pinch-points to define them.

The contact of two spheres x, y is expressed by the vanishing of the polar of x with regard to y, $\sum_{i=1}^{6} \frac{\partial \Pi}{\partial x_i} y_i = 0$. In particular, two consecutive spheres of the surface, x and $x + dx$ will be in contact if $\Pi(dx) = 0$. If the functions which define x are rational and of degree n, there are $2(n - 2)$ pinch-points on the surface. If $\Pi\,(dx)$ vanishes identically, the surface reduces to a curve enveloped by spheres.

Now, suppose all of the spheres x to belong to a linear complex of spheres, $\psi = 0$. The spheres which touch x and belong to ψ must touch x in points of its trajectory circle with regard to ψ; this circle lies on the fundamental sphere s of ψ. Similarly, for the consecutive sphere $x + dx$ of the surface. The two corresponding trajectory circles intersect in two points P_1, P_2 which lie also on c. From these two points proceed two tangent pencils of spheres which belong to ψ and touch the spheres x, $x + dx$. The locus of each of these points is a line of curvature on the surface, for each locus lies on a sphere which everywhere cuts the surface at a constant angle; hence: *Every annular surface contained in a linear spherical complex has a line of curvature determined by the intersection of the fundamental sphere of the complex with the surface.* This curve can be found by rational operations. When a certain linear condition exists among the coefficients of ψ, this line of curvature lies in a plane.

Now, suppose that the spheres which envelop the surface belong to two linear complexes, ψ_1, ψ_2; then they belong to every complex of the system

$$\psi_1 + \lambda\psi_2 = 0,$$

and all the lines of curvature become rational, being the curve of intersection of the surface with the fundamental spheres $s_1 + \lambda s_2$. Among these complexes are two special ones, whose fundamental spheres s_1', s_2' are touched by all the spheres of the congruence. These two spheres touch the surface along lines of curvature. Their centres are nodal points on the surface of centres of the annular surface. The two directrices s_2', s_2' intersect in a circle $\varkappa$ which lies on every fundamental sphere $s_1 + \lambda s_2$. Among the fundamental spheres is one whose radius becomes infinite; it is the plane ϖ of the circle $\varkappa$.

All the lines of curvature of an annular surface containing two spherical directrices are determined by rational operations; every such surface must have one line of curvature of the second system which is a plane curve.

To see the relation of these lines of curvature, it is easiest (though not necessary) to interpret the properties of asymptotic lines of ruled surfaces having two rectilinear directrices by means of Lie's "Abbildung" [see vol. V of the Bulletin of the Amer. Math. Soc., p. 343]. The following theorems then result:

Each fundamental sphere $s_1 + \lambda s_2$ cuts every circle of curvature c in two points:

as λ varies, these points describe an involution on c, whose double points are the points of contact of c with s_1', s_2'.

All the lines of curvature of the second system touch each other at the pinch-points.

A simple geometrical construction shows that the locus of the centre of this involution is a curve lying wholly in the plane ϖ. Moreover, there are a number ($2n$) of point spheres belonging to the annular surface; these points all lie on $\varkappa$; the locus in question passes through each of these points, and as no centre can lie within $\varkappa$, the locus touches $\varkappa$ in $2n$ points. Its order is thus $2n$.

If two planes ϖ_1, ϖ_2 are contained in the pencil $s_1 + \lambda s_2$ of fundamental spheres, then all become planes, $\varkappa$ becomes a straight line and all the lines of curvature are plane. The locus of the centre of the involution on c is the line $\varkappa$, axis of the pencil of planes.

If an annular surface has two plane directors, all of its lines of curvature are plane.

Surfaces of revolution are included in this category; the line $\varkappa$ is now the axis of revolution, and the double points of the involution on c are imaginary.

The simplest type of this form of surface is that of Dupin's cyclides. The line $\varkappa$ is the line joining the nodes.*

Now, suppose ψ_1 is the linear complex composed of the planes of space, its fundamental sphere is the plane at infinity. If ψ_2 is general, all the other fundamental spheres $s_1 + \varkappa s_2$ are concentric with the directrix s_2'. As all of the generators of the annular surface are now planes, it becomes a developable; all the generating planes of this developable touch s_2'. The theorems regarding lines of curvature hold the same as before, hence:

If the planes which generate a developable surface all touch a fixed sphere, its (non-linear) lines of curvature are determined by its intersection with a family of spheres concentric with the fixed one.

An immediate deduction from this theorem is the following: If one line of curvature of a developable is a spherical curve, they all are, and the spheres are all concentric. In this form it was obtained by Paul Serret by means† of differential geometry.

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*The second pair of nodes (always imaginary) is to be entirely ignored, as they belong to the second generation of the cyclide, and are not included in the envelope of spheres which define the surface.

† Théorie géométrique et méchanique des lignes de double courbure. Paris, 1860.

In particular, if the radius of s_2' becomes zero, all the planes pass through a point and the developable surface becomes a cone. The theorem then states the simple fact that the lines of curvature of a cone are determined by its lines of intersection with a family of spheres concentric with its vertex.

If ψ_1 is composed of the points of space, the annular surface becomes the curve wherein s_2 intersects the point locus of the third complex which defines the surface.

Finally, if ψ_1 is points, ψ_2 planes, the annular surface reduces to the line of curvature enveloped by minimum lines on the point locus.

When the two directrices coincide, a particularly interesting case is presented; now the involution becomes degraded as the double points coincide; one point of every pair of conjugates coincides with this double point, and every fundamental sphere contains the curve of contact with the directrix. It is, therefore, a circle, and evidently the circle $\varkappa$. The sphere s' is now itself usually a generator. The other fundamental spheres cut the surface in $\varkappa$ and in one other line of curvature. The first theorem for the general case has now lost its meaning. The locus of the centre of involution is now the circle $\varkappa$. The circle $\varkappa$ may become a straight line. The necessary modifications of the theorem for this case are obvious.

Analytically, one may proceed thus:

If the surface is defined by six equations of the form

$$x_i = \phi_i(t),$$

these values x_i are to be substituted in an equation of the form

$$\sum_{i=1}^{6} a_i x_i = 0,$$

and the a_i so determined that the equation

$$\sum a_i \phi_i = 0$$

is identically satisfied. When the degree of ϕ_i is 2, there are three independent complexes: the surface is a Dupin's cyclide. When ϕ_i is of degree 3, there are 2 independent complexes. The resulting surface is of order six (or five, and the plane at infinity). *Every annular surface of order* 6 *has only rational lines of curvature.* When ϕ_i is of degree 4, the surface belongs to a single linear complex (in general). The other lines of curvature can be determined by the method outlined by Picard.

If the surface is defined by complexes, there are four equations of condition

$$f(x)=0, \quad \psi_1(x)=0, \quad \psi_2(x)=0, \quad \Pi(x)=0$$

among the homogeneous parameters $x_1 \ldots . x_6$ of the equation

$$\lambda_0(x^2+y^2+z^2)-2\lambda_1 x-2\lambda_2 y-2\lambda_3 z+\lambda_4=0,$$

wherein the λ_i are all linear and homogeneous in $x_1 \ldots . x_6$, and the envelope of the sphere is required.

The determination of the point equation of the fundamental sphere of the complex

$$\psi_1+\varkappa\psi_2=0$$

has been considered in a previous paper.*

These properties are sufficient to make a theorem which I proved some time ago more definite.† The theorem can now be completed by the statement:

In the latter case the surface is an annular surface whose generators are included among those that cut the fixed sphere at the constant angle.

CORNELL UNIVERSITY, *Sept.* 20, 1898.

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* V. Snyder, "Determination of Nodes in Dupin's Cyclides." Annals of Mathematics, vol. XI, p. 137.

† V. Snyder, "Geometry of some Differential Expressions in Hexaspherical Coordinates." Bulletin Amer. Math. Society, vol. IV, p. 147, middle of page.